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The Bounded Weak-Star Topology and the Bounded Analytic Functions*

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INTRODUCTION

We denote by $B_H(G)$ the space of bounded analytic functions on the plane region G and say that a sequence $\{f_n\}$ of elements of this space is boundedly convergent to f if and only if (i) $|f_n(z)| \leqslant M < \infty$ for all $n = 1, 2, 3,\ldots$ and all $z \in G$; (ii) $f_n(z) \to f(z)$ for each $z \in G$. It is natural to consider the strongest topology τ on $B_H(G)$ in which $\{f_n\}$ converges to f in τ if and only if $\{f_n\}$ converges boundedly to f. We show that $\tau = \beta$, where β is the strict topology introduced by Buck [2] and extensively studied in the context of bounded analytic functions by Rubel and Shields [19]. To some extent the present paper is a continuation of [19]. Further, we show that $\beta = \gamma$, where γ is the bounded weak-star topology on $H^\infty(G)$ considered as the dual of a certain quotient of a space of measures studied in [19]. Here,

* Since this paper was written, the following paper, which is related to the first theorem of section 2 of the present paper, has come to our attention: J. R. DORROH, The localization of the strict topology via bounded sets. *Proc. Amer. Math. Soc.* **20** (1969), 413–414.

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$H^\infty(G)$ denotes the Banach algebra composed of $B_H(G)$ with the norm $\|f\| = \sup\{|f(z)| : z \in G\}$. From this identification of β with γ, it follows that an arbitrary subset of $\beta(G)$ is closed if and only if it is sequentially closed, even though $\beta(G)$ is not metric, a result first proved by Hessler, but never published. We prove that the dual, in the weak-star topology α, of an infinite-dimensional Banach space never has this sequential closure property, thus giving yet another proof that $\alpha(G) \neq \beta(G)$, where $\alpha(G)$ is $H^\infty(G)$ with its weak-star topology α. We also consider the question: Does β coincide with the Mackey topology m on $H^\infty(G)$? This was answered in the negative by Conway in [3] for the special case $G = D$, the open unit disc. We extend the negative answer to a larger class of regions G, but do not obtain the complete answer. In the process, we raise the question about the existence of inner (or at least hypo-inner) functions in the unit disc that are automorphic under the group of covering transformations, and answer it satisfactorily for our needs in case G is the complement of a closed set of positive linear measure that lies on the real axis. This is related to the work of Ahlfors, Bishop, Royden, and others on extremal bounded analytic functions.

Some natural questions regarding topologies on the bounded analytic functions are better viewed in a general Banach space setting. In the concluding section, we prove some relevant, general results on the weak-star and bounded weak-star topologies.

The reader is referred to the bibliography, where some articles are listed that have a close relation to this paper even though they are not specifically referred to.

1. The Topologies

In Kisyński [16] (see also Dudley [7]), it is shown that a topology T is defined on a set X if a notion of "convergence" of a sequence, $x_n \to x$, is given, for which

(i) For each $x \in X$, the constant sequence $x_n = x$ for $n = 1, 2, \ldots$ satisfies $x_n \to x$.

(ii) If $x_n \to x$ and $\{y_m\}$ is any subsequence $\{x_{n_m}\}$ of x_n then $y_m \to x$.

(iii) If $x_n \to x$ and $x_n \to x'$, then $x = x'$.

(iv) If $x_n \not\to x$ (i.e., it is false that $x_n \to x$) then there is a subsequence $\{y_m\}$ of $\{x_n\}$ such that for any subsequence $\{z_p\}$ of $\{y_m\}$, $z_p \not\to x$.

The topology T is defined on X by choosing, for open sets, those sets U in X with the property that if $x_n \to x$ and $x \in U$, then all but finitely many of the x_n belong to U. Kisyński proved in [16] that a sequence $\{x_n\}$ converges in the topology T to x if and only if $x_n \to x$. It is easy to see that T is the strongest topology with this property. Choosing $X = B_H(G)$ and convergence as bounded convergence, we denote the resulting topology by τ.

To describe the strict topology β, we let $C_0(G)$ be the space of continuous functions on G that tend to 0 as we leave compact subsets of G. If $k \in C_0(G)$, then $\|f\|_k = \sup\{|f(z)\, k(z)| : z \in G\}$ defines a seminorm on the space $B_H(G)$. As k runs through $C_0(G)$, the corresponding seminorms define a locally convex topology on $B_H(G)$ called the strict topology β. It was proved in [19] that a sequence $\{f_n\}$ converges β to f if and only if $\{f_n\}$ converges boundedly to f. Indeed, the β topology, restricted to bounded subsets of $H^\infty(G)$, coincides with the topology of uniform convergence on compact subsets of G.

We now describe some other topologies on $B_H(G)$. This space becomes a familiar Banach algebra $H^\infty(G)$ when equipped with the supremum norm. Let $M(G)$ denote the Banach space of bounded complex Borel measures μ that have all their mass in G, under the total variation norm $\|\mu\| = \int d\,|\mu|$. There is a duality $\langle B_H(G), M(G)\rangle$ defined by $\langle f, \mu\rangle = \int f\, d\mu$. Let $N = \{\mu \in M(G) : \int f\, d\mu = 0, \forall f \in B_H(G)\}$ be the annihilator of $B_H(G)$ in $M(G)$, and let $M'(G) = M(G)/N$ be the quotient Banach space. It was proved in [19] that $H^\infty(G)$ is the dual Banach space of $M'(G)$ and that $M'(G)$ is separable. We denote by $[\mu]$ the equivalence class that μ belongs to. We further denote by α the weak-star topology on $H^\infty(G)$. Thus, a generalized sequence $\{f_\gamma\}$, $\gamma \in \Gamma$, converges to f in $\alpha(G)$ if and only if $\int f_\gamma\, d\mu$ converges to $\int f\, d\mu$ for each $\mu \in M(G)$. It was shown in [19] that $\alpha(G)$ has the same convergent sequences (to the same limits) as $\beta(G)$, that the α-bounded and β-bounded sets are both just the norm-bounded sets, and that the α topology, when restricted to bounded subsets, coincides with the topology of uniform convergence on compact subsets of G.

Let us now consider the topology γ, defined as the bounded weak-star topology on $H^\infty(G)$, as the dual of $M'(G)$. This topology is defined as the strongest topology that agrees with the weak-star topology on bounded subsets of $H^\infty(G)$. In γ, a subset is closed if and only if its intersection with each dilation of the closed unit ball is weak-star closed. It is characterized [20, p. 150] as the topology of uniform convergence on strongly compact subsets of the pre-dual, or equivalently [8, p. 427] as the topology of uniform convergence on sequences in the pre-dual that tend strongly to 0.

Finally, let us consider the Mackey topology m on $H^\infty(G)$, defined as the strongest locally convex topology on $H^\infty(G)$ that has the same dual space as $\alpha(G)$, namely $M'(G)$. The topology m is characterized [20, p. 131] as the topology of uniform convergence on all convex and circled subsets of $M'(G)$ that are weakly compact.

2. Comparison of the Topologies

In [19], it was shown that $\alpha \subseteq \beta \subseteq m$ and that $\alpha \neq \beta$ if $B_H(G)$ contains a nonconstant function.

THEOREM. *The strict topology on $B_H(G)$ is just the bounded weak-star topology. That is, $\beta = \gamma$.*

Proof. Since the β topology agrees with the α topology on bounded sets, it follows that $\beta \subseteq \gamma$. This says, in effect, that given $k \in C_0(G)$, there exists a null sequence $\{[\mu_n]\}$ of elements of $M'(G)$ such that $|\int f\, d\mu_n| \leqslant 1$, $n = 1, 2, 3, \ldots$ implies $\sup\{|f(z)\, k(z)| : z \in G\} \leqslant 1$. A direct proof of this can easily be supplied by the reader. The converse, that $\gamma \subseteq \beta$, reduces to proving that, given a norm-null sequence $\{[\mu_n]\}$ of nonzero elements of $M'(G)$, there exists a $k \in C_0(G)$ such that $\sup\{|f(z)\, k(z)| : z \in G\} < 1$ implies $|\int f\, d\mu_n| < 1$ for $n = 1, 2, 3, \ldots$. We are grateful to *E.* Speer for simplifying our original proof of this fact. Let C_m be a sequence of compact subsets of G with $C_m \subsetneq C_{m+1}$ and $\bigcup C_m = G$, chosen so that

$$|\mu_n|(G \backslash C_m) \leqslant 2^{-m} \qquad \text{for} \quad n \leqslant m.$$

Now we take $k \in C_0(G)$ so that $k(z) \geqslant 4 \max(\|\mu_m\|, 1/m)$ for $z \in C_m$. Then if $|fk| < 1$ on G, we have

$$\left|\int_G f\, d\mu_n\right| < \int \frac{d|\mu_n|}{k} = \left\{\int_{C_n} + \int_{C_{n+1}/C_n} + \int_{C_{n+2}/C_{n+1}} + \cdots\right\} \frac{d|\mu_n|}{k}$$

$$\leqslant \int_{C_n} \frac{d|\mu_n|}{4\|\mu_n\|} + \sum_{m=n}^{\infty} \int_{C_{m+1}/C_m} \frac{(m+1)}{4}\, d|\mu_n|$$

$$\leqslant \frac{1}{4}\left(1 + \sum_{m=n}^{\infty} \frac{m+1}{2^m}\right) \leqslant 1.$$

It is now relevant to produce the following result, which seems to be "folklore". We remark that the weak-star convergent sequences coincide with the bounded weak-star convergent sequences.

For the remainder of this paper we shall adopt the following notations. By B we will denote a Banach space, and B^* will denote its dual. On B^* we will consider several topologies. First is the weak-star topology denoted by α. Then there is the bounded weak-star topology denoted by γ. Then there is the Mackey topology denoted by m. Finally there is the topology τ that is defined as the strongest topology that has the same convergent sequences (to the same limits) as α.

PROPOSITION. *Let B be a separable Banach space with dual B^*. A subset F of B^* is closed in the bounded weak-star topology γ on B^* if and only if it is sequentially closed.*

Proof. It is clear that a γ-closed subset must be sequentially closed. For the converse, suppose that F is sequentially closed and let, for $a > 0$, $S_a = \{x^* \in B^* : \| x^* \| \leqslant a\}$. Then S_a is α-closed (where α is the weak-star topology on B^*), and since B is separable, the relative α topology of $F \cap S_a$ is metrizable [8, p. 426]. Hence $F \cap S_a$ is α-closed, and it follows that F is γ-closed. The next result follows directly.

THEOREM. *A subset of $\beta(G)$ is closed if and only if it is sequentially closed.*

This result was first proved by P. Hessler by entirely different methods. The corresponding result for *convex* subsets of $\beta(G)$ was used extensively in [19].

The next result follows directly from the work of S. P. Franklin [10]. We recall that a map $\varphi : X \to Y$ is called a quotient map if $\varphi^{-1}(A)$ is open precisely when A is.

COROLLARY. *Although the bounded weak-star topology γ on B^* is not metrizable, (B^*, γ) is the image of a metric space under a quotient map, if B is separable.*

Problem. Can the metric space above be chosen to be a linear metric space and the quotient map to be also a linear homomorphism?

THEOREM. *The bounded weak-star topology on the dual B^* of a* *separable Banach space B is the strongest topology on B^* in which $x_n^* \to x^*$ if and only if x_n^* converges to x^* in the weak-star topology.*

COROLLARY. $\beta = \tau$, *that is, the strict topology is the strongest topology on $B_H(G)$ in which $f_n \to f$ if and only if $\{f_n\}$ converges boundedly to f.*

Problem. In those cases where $\beta \neq m$ (see later), what are the m-convergent sequences of bounded analytic functions?

Proof of Theorem. We easily see that $\gamma \subseteq \tau$. To prove that $\tau \subseteq \gamma$, we must prove that every τ-closed subset is γ-closed. But every τ-closed subset F is τ-sequentially closed. It follows from the first remarks on topologies associated with a mode of sequential convergence that the τ-convergent sequences are just the weak-star convergent sequences. Thus F is γ-sequentially closed and hence F is γ-closed.

3. NONCOINCIDENCE OF THE STRICT AND MACKEY TOPOLOGIES

As we show in the concluding section of this paper, $\beta \neq m$ if and only if there is a weakly compact set in $M'(G)$ that is not strongly compact.

In this section, we prove that for several classes of regions G, the strict topology β on $B_H(G)$ does not coincide with the Mackey topology m. It seems likely that this is always the case when $B_H(G)$ contains a nonconstant function, but we have no proof, and suspect that a general proof would be difficult to find. To begin with, we recall that a continuum C in the boundary of a region G is called a free boundary component of G if some neighborhood of C intersects no other boundary components of G.

THEOREM. *If G has a free boundary component C then the strict topology and the Mackey topology on $B_H(G)$ are distinct.*

Proof. The component of $\mathbb{C} \setminus C$ containing G can be mapped conformally onto the open unit disc D. The image of G will be a subregion of the disc that contains an annulus A with center at the origin, whose outer boundary is the unit circumference. We shall assume that G was of this form initially. Each $f \in B_H(G)$ may be expressed in a unique way as $f = f_1 + f_2$ where f_1 is analytic for $|z| < 1$, f_2 is analytic for $|z| > r$, where r is the inner radius of A, and $f_2(\infty) = 0$. If $r < \rho < 1$, then f_2 is bounded for $|z| \geqslant \rho$ so that f_1 will be bounded for $|z| < 1$. We identify f_1 with its radial boundary values on the unit circumference and define

$$L_n(f) = \int_{-\pi}^{\pi} f_1(e^{i\theta})\, e^{-in\theta}\, d\theta = \int_{-\pi}^{\pi} [f_1(e^{i\theta}) + f_2(e^{i\theta})]\, e^{-in\theta}\, d\theta$$

for $n = 1, 2, 3, \ldots$. Each L_n represents an element of $M'(G)$. Indeed, we have

$$L_n(f) = \int f\, d\mu_n$$

where $d\mu_n = z^{-(n+1)}\, dz$ on the curve $z = \rho e^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$ and $r < \rho < 1$. By the Riemann-Lebesgue lemma, $L_n(f) \to 0$ for each $f \in B_H(G)$. The subset of $M'(G)$, determined by the L_n together with 0, is therefore weakly compact. This set cannot be norm-compact since the norm of each nonzero element is 2π, as may be seen by choosing $f(z) = z^n$.

Another possible method of showing that $\beta \neq m$ involves the identification of $B_H(G)$ with a certain algebra of bounded automorphic functions on the unit disc. If the region G admits nonconstant bounded holomorphic functions, its universal covering surface can be taken to be the unit disc. This simply means that there exists a certain holomorphic function φ, defined for $|z| < 1$, such that (i) φ maps the unit disc onto G, (ii) φ' never vanishes, and (iii) direct analytic continuation of φ^{-1} is always possible along any arc in G.

Any such map φ is called a covering map. Associated with it is a group Γ, the covering group, of conformal one-to-one maps T of the unit disc onto itself, defined by: $T \in \Gamma$ if and only if $\varphi(Tz) = \varphi(z)$ whenever $|z| < 1$. Each $f \in B_H(G)$ is then identified with an element g of $B_H(D)$ according to the rule $g = f \cdot \varphi$. The function g is automorphic in that $g(Tz) = g(z)$ for all $T \in \Gamma$ and all $z \in D$.

Furthermore, each bounded automorphic function is of the form $f \circ \varphi$ for some $f \in B_H(G)$. We denote by B_Γ the space of all bounded analytic automorphic functions with respect to the covering group Γ. We have just described how $B_H(G)$ may be identified with B_Γ. As bounded analytic functions, the elements of B_Γ will have associated radial boundary values on the unit circumference that are also invariant under the covering group: $g(e^{i\theta}) = g(Te^{i\theta})$ for each $T \in \Gamma$.

DEFINITION. A nonconstant analytic function f defined in the unit disc D will be called a hypo-inner function if $|f(z)| \leqslant 1$ for all $z \in D$ and $|f(e^{i\theta})| = 1$ on a set of positive measure in $\{z : |z| = 1\}$.

THEOREM. *Assume that B_Γ contains a hypo-inner function, where Γ is the covering group of the region G. Then the strict topology on $B_H(G)$ is different from the Mackey topology.*

Remark. In particular, if B_Γ contains an inner function f ($|f(e^{i\theta})| = 1$ almost everywhere), as it will [17] when G is finitely connected, then the theorem applies. The case of finite connectivity, however, is already handled by the preceding theorem. In general, it appears that the following function f is an inner function, but we cannot prove it: we fix two points a and b in D such that $a \neq Tb$ for all $T \in \Gamma$, and choose $f \in B_\Gamma$ subject to the constraints $f(a) = 0$, $\|f\|_\infty \leqslant 1$, so as to maximize $|f(b)|$. See [9] for some interesting results along these lines.

Proof of the Theorem. Let f be a hypo-inner function in B_Γ. The sequence of functions $\{f^n\}$, $n = 1, 2, 3,...$ converges boundedly to 0 in D. Considering these functions as elements of the unit ball of H^2, we choose and reindex a weakly convergent subsequence. It is clear that this new sequence must have 0 as its weak limit, as its extension to D will vanish everywhere by the Cauchy integral formula. Thus,

$$L_n(g) = \int_{-\pi}^{\pi} g\bar{f}^n \, d\theta \to 0 \qquad \text{as} \quad n \to \infty$$

for each $g \in B_\Gamma$. Furthermore, each L_n is a continuous linear functional, since the kernel of each L_n is weak-star closed. To see this, let $g_m \to g$ be a boundedly convergent sequence in B_Γ for which $L_n(g_m) = 0$ for $m = 1, 2, 3,...$. Then $g \in B_\Gamma$. As before, we pass to an H^2 weakly convergent subsequence, that must converge to g, and obtain

$$0 = L_n(g_m) = \int g_m \bar{f}^n \, d\theta \to \int g\bar{f}^n \, d\theta = L_n(g).$$

Since a sequentially weak-star closed convex subset is necessarily weak-star closed, the continuity of L_n is established. But we also have $\|L_n\| \geqslant L_n(f^n) \geqslant \epsilon > 0$ so that the L_n, together with 0, form a weakly compact subset of $M'(G)$ that is not norm-compact.

4. DENJOY REGIONS AND ORTHOCIRCULAR DOMAINS

We now construct regions for which the preceding theorem applies whereas the earlier one does not. The regions we shall consider are the so-called Denjoy regions that we define as those obtained by deleting a closed linear set of positive measure from the plane. Denjoy [6] has shown that the complement of a closed linear set will support nonconstant bounded analytic functions if and only if the set has positive linear measure. The main result of this section is that if G is a Denjoy region, then B_Γ contains a hypo-inner function, so that $\gamma \neq m$.

We now describe what shall be called orthocircular domains. We begin by choosing an arbitrary closed subset F of the unit circumference, $\{z : |z| = 1\}$, that has positive measure. The complement of F in the circle will be a collection $\{J_n\}$ of disjoint open intervals. Corresponding to each J_n, we construct the circle orthogonal to the unit circle which passes through the endpoints of J_n. We denote by γ_n the arc of that circle that intersects $\{z : |z| \leqslant 1\}$. Then the $\{\gamma_n\}$ together with F bound a simply connected (indeed star-shaped) region Ω with rectifiable boundary. Any such region Ω in D bounded by nonintersecting circular arcs orthogonal to the unit circle, together with the remaining points of the unit circle, will be called an *orthocircular domain.* These domains enter into the representation of the unit disc as the universal covering surface of a Denjoy region, as we shall later show.

We wish to thank A. Beurling for suggesting the following construction. Let Ω be an orthocircular domain and let φ_0 be a conformal map of Ω onto the upper half-plane. From the classical theory of such maps, we know that φ_0 extends to a continuous function on the closure of Ω. We denote the extension by φ. Since the boundary of Ω is rectifiable, an easy modification of a theorem of F. Riesz [21, p. 318] guarantees that along the boundary of Ω, φ preserves sets of positive measure and sets of zero measure. Consequently, the image of F will be a closed subset E of positive measure of the real line $\mathbb{R}$, and each γ_n will map onto a complementary open interval I_n in $\mathbb{R}$. A reflection across γ_n yields a mapping of the reflected region onto the lower half-plane that can be extended analytically across γ_n. Reflection across any γ_n extends φ to an analytic mapping of some new, larger, orthocircular domain onto the complement of the closed set E. This new region Ω' is a fundamental region for the Denjoy region $\mathbb{C}\setminus E$, and φ is the covering map of that region. We may continue reflecting and extending the domain of φ until the unit disc is exhausted. This may be proved directly as follows. Assume, by way of contradiction, that some point $z \in D$ is not in the limit region. Then each Ω_n (the region obtained after n reflections in all the γ_k) omits z, and there will exist a sequence of orthocircular boundary arcs of the Ω_n that separate z from Ω_n. These arcs converge to an orthocircular limiting arc γ_0. The set U of points enclosed by γ_0 and the unit circle is disjoint from the limit region. This region is invariant under all reflections over the boundary arcs $\gamma^{(j)}$ of any Ω_n. In particular, one may choose arcs $\gamma^{(j)}$ so close to γ_0 that a reflection in $\gamma^{(j)}$ will carry points from the limit region into U. Perhaps the best way to see this is to transfer the problem to the upper half-plane. Then points with large imaginary part, that lie above the center of the limiting arc, will be carried via reflection to points near the real line, giving a contradiction.

We now set

$$f(w) = \int_E \frac{dx}{x - w},$$

and note that f is analytic in $\mathbb{C}\setminus E$ and maps this region into the horizontal strip bounded by $y = \pm\pi$. Moreover, the imaginary part of f has the nontangential limit $\pm\pi$ at almost all points of E. The ambiguity in sign arises at those points of E where the limit is taken in the upper or lower half-plane, respectively. Since $\exp f = \exp(u + iv)$ maps from the upper half-plane into itself, it too has nontangential limits almost everywhere, and from this we conclude that f itself also has this property. We denote by E_1 the subset of E where f has nontangential limits. For $x \in E_1$, we let $e^{i\theta} = \varphi^{-1}(x)$ and suppose that Δ is an arc that approaches x vertically. Then $\delta = \varphi_0^{-1}(\Delta)$ is a simple arc in Ω that terminates at $e^{i\theta}$. Therefore, for almost all $e^{i\theta} \in F$, there exists an arc δ in Ω terminating at $e^{i\theta}$ such that

$$\lim_{\delta} \operatorname{Im}(f \circ \varphi) = \pi.$$

The map

$$H(\zeta) = (e^{\zeta/2} - 1/e^{\zeta/2} + 1)$$

carries the above-described horizontal strip onto the unit disc. The composite $g = H \circ f \circ \varphi$ then has radial limits almost everywhere on the unit circle. Furthermore, at almost every point of F, there is an arc terminating at that point, along which g has a limit of modulus 1. A theorem of Lindelöf [17] tells us that the radial limit of g at this point is the same as its limit along the arc. Thus $|g| \to 1$ nontangentially at almost every point of F. The function φ, and hence g, will be automorphic with respect to the group Γ generated by the products of pairs of reflections. This group is the covering group of the Denjoy region $\mathbb{C}\setminus E$.

We conclude the section by establishing a correspondence between

Denjoy regions and orthocircular domains, from which the next result follows immediately after the considerations of the first part of this section.

THEOREM *If G is a Denjoy region then there exists a hypo-inner analytic function on G.*

Resuming our discussion, we let E be a closed subset of $\mathbb{R}$ that has positive measure, and suppose for simplicity that $\infty \in E$. Write $\mathbb{R}\setminus E = \bigcup (a_i, b_i)$ as a union of disjoint open intervals. There exists a conformal map h_n that transforms the region $\mathbb{C}\setminus\bigcup_{i=1}^{n} [a_i, b_i]$ into a region bounded by n disjoint circles [21, p. 424]. By composing with a suitable Möbius transformation, we may assume that the circle corresponding to $[a_1, b_1]$ is the unit circle and that $h_n(a_1) = -1$ and $h_n(b_1) = 1$. When this normalization is made, all the remaining circles will then be centered on the real axis, as we shall now show. We wish to thank Z. Nehari for calling the following proof of this fact to our attention.

We know [21, p. 426] that any other conformal map that carries $\mathbb{C}\setminus\bigcup_{i=1}^{n} [a_i, b_i]$ onto a circular region (i.e., one bounded by n circles) differs from h_n by a Möbius transformation. In particular, we must have

$$\bar{h}_n(\bar{z}) = \frac{h_n(z) + \alpha}{1 + \alpha h_n(z)} = k_n(z),$$

where α is real and $|\alpha| < 1$, since the unit circle and ± 1 are preserved. The range of k_n is obtained by conjugating the range of h_n, whereas $g(w) = (w + \alpha)(1 + \alpha w)^{-1}$ transforms the upper and lower half-planes into themselves. If h_n maps the (doubly covered) interval $[a, b]$ onto the circle C, then k_n maps the interval onto the circle conjugate to C. If C lay in the upper or lower half-plane, then $k_n = g \circ h_n$ would preserve this property, and we would have a contradiction. Similarly, if C did not intersect the real axis orthogonally, then by conformality, k_n would map C onto a circle with the same angle of intersection with the real axis. But this image circle must coincide with the circle conjugate to C, which again yields a contradiction. From this, we see that h_n does indeed map $\mathbb{C}\setminus\bigcup_{i=1}^{n} [a_i, b_i]$ onto the exterior of n discs whose centers are on the real axis. Moreover, these discs are preserved by g, whose only fixed points are ± 1, and therefore $\alpha = 0$ so that $\bar{h}_n(\bar{z}) = h_n(z)$. Thus we see that h_n (or $-h_n$) carries the upper half-plane onto the subregion of the upper half-plane bounded by n disjoint semicircles orthogonal to $\mathbb{R}$, together with the remainder of $\mathbb{R}$. A conformal map of the upper half-plane onto the unit disc carries this region onto an orthocircular domain.

The remainder of the argument becomes technically simpler if we map the upper half-plane conformally onto the unit disc so that the intervals (a_m, b_m) complementary to the set E go over to disjoint open arcs I_m on the unit circle, and E maps onto the closed complementary set F. We have established the existence of conformal maps f_n of the unit disc onto orthocircular domains Ω_n, with the property that $I_1, I_2, \ldots, I_n$ go over to circular arcs $\delta_1^{(n)}, \delta_2^{(n)}, \ldots, \delta_n^{(n)}$ that are orthogonal to the unit circle for each $n = 1, 2, 3, \ldots$.

By means of a Möbius transformation, we may assume that each $\delta_1^{(n)}$ is the same arc δ_1 whose end points are $\exp(\pm i\pi/4)$ and that no other arc has end points with smaller arguments. Thus, δ_1 does not "surround" the remaining $\delta_j^{(n)}$, $2 \leqslant j \leqslant n$.

The maps f_n extend to continuous univalent functions on the closed unit disc, and, by a theorem of F. and M. Riesz [21, p. 318], the restrictions $f_n(e^{i\theta})$ of these functions to the boundary are absolutely continuous functions of θ. Their total variation, $\int_{-\pi}^{\pi} |f_n'(e^{i\theta})| \, d\theta$, which is the arc length of the boundary of the range, remains bounded as $n \to \infty$. An explicit estimate of how large the arc length can be, while interesting, is simple and will be left to the reader. A well-known theorem of Helly about functions of bounded variation implies that a subsequence of $\{f_n\}$ may be selected that converges to a function f at each point $e^{i\theta}$ of the unit circle, and f will have bounded variation. Passing to a subsequence and reindexing, we may assume that $\{f_n\}$ also converges uniformly on compact subsets of the open unit disc to an analytic function $\tilde{f}$. This is the function we seek.

The negative Fourier coefficients of f all vanish so that $f \in H^\infty$ and we have the limits

$$f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} P_z * f_n = P_z * f,$$

where P_z is the Poisson kernel for the point z. Thus f coincides with the boundary values of $\tilde{f}$ and we may write $f(z)$ in place of $\tilde{f}(z)$. We shall show in a moment that f is continuous on the closed unit disc. If $f(z) = c$ for all $|z| \leqslant 1$ and α and β are the end points of I_1, we obtain

$$c = \lim_{n\to\infty} f_n(\alpha) = \lim_{n\to\infty} f_n(\beta).$$

But for each n, $|f_n(\alpha) - f_n(\beta)| = \sqrt{2}$, which is a contradiction, so that f cannot be a constant. Therefore f is univalent.

To prove that f is continuous, we observe that because each f_n is absolutely continuous on the unit circle, its derivative f_n' belongs to

H^1 and $\|f_n'\|_1 \leqslant M$ for some $M > 0$ and all $n = 1, 2, 3,\ldots$. Now H^1 is the dual space of C/A_0 where C is the algebra of all continuous functions on the unit circle and A_0 is the subalgebra of those functions that are analytic inside the unit disc and vanish at the origin. Passing to a subsequence, if necessary, we assume that $\{f_n'\}$ converges to a function $h \in H^1$ in the weak-star sense. It is easy to verify that $f'(z) = P_z * h$, so that $f' \in H^1$. A theorem of F. Riesz [12, p. 360] implies that, under these circumstances, f is continuous on the closed unit disc and absolutely continuous on the circumference.

Let α_n and β_n be the end points of the interval I_n . We construct the circular arc δ_n in $\{w : |w| \leqslant 1\}$ orthogonal to $\{w : |w| = 1\}$, with end points $f(\alpha_n)$ and $f(\beta_n)$. Such an arc cannot degenerate to a point. Indeed, if $f(\alpha_n) = f(\beta_n)$, then as $m \to \infty$ all the points along the arc $f_m(I_n)$ would tend to this single value. That is, f would be constant along I_n and thus would reduce to a constant everywhere in the disc, which we have shown to be impossible.

The arcs δ_n together with the remainder of the unit circle bound an orthocircular domain Ω. The components of the complement of Ω in the open unit disc will be called shaded regions, to aid in visualizing the argument which follows.

No interior point of a shaded region belongs to the range of f. To see this, note first that by our construction of Ω, the end points of δ_n are the limits of the end points of the arcs $\delta_n^{(m)}$ as $m \to \infty$. As these arcs lie on circles orthogonal to the unit circle, they must converge uniformly to δ_n in the sense that their centers and radii converge to the center and radius of the circle which describes δ_n . From this, it should be clear that any point in a shaded region is eventually omitted by the Ω_m and therefore cannot be in the range of f.

Finally, suppose that f omits some point $w \in \Omega$. Then the range of f contains a boundary point w_0 which does not lie on the boundary of Ω, with $|w_0| < 1$. Since f is continuous in $\{|z| \leqslant 1\}$, we choose $e^{i\theta}$ so that $f(e^{i\theta}) = w_0$. Then $e^{i\theta} \notin F$ since $|f| = 1$ at all points of F. Therefore $e^{i\theta}$ belongs to some I_n . Since $f_m(e^{i\theta}) \to f(e^{i\theta})$ and $f_m(e^{i\theta}) \in \delta_n^{(m)}$, it follows by our previous remark about the convergence of $\delta_n^{(m)}$ to δ_n that we must have $w_0 \in \delta_m$. Therefore the range of f is Ω and the proof is complete.

3. Some Results on Functional Analysis

We prove here some general results on the weak-star and bounded weak-star topologies. They specialize, in case $B = M'(G)$, to results about the α and β topologies on $B_H(G)$, giving new proofs of some of the results in [19].

PROPOSITION. *Suppose B is an infinite-dimensional Banach space, and B^* its dual. Then B^* in the weak-star topology is not complete.*

Proof. Let L be a discontinuous linear functional on B. Order the finite subsets of B by inclusion to get a directed set Λ. For each finite subset $\lambda = (x_1, x_2, ..., x_n)$, choose a continuous linear functional L_λ such that $L_\lambda(x_j) = L(x_j)$, $j = 1, 2, ..., n$. The net $\{L_\lambda\}$, $\lambda \in \Lambda$, is a Cauchy net that is not weak-star convergent.

PROPOSITION. *Let B be a Banach space, and let B^* be its dual space. Then B^* in the bounded weak-star topology is complete.*

Proof. Let $\{L_\lambda\}$, $\lambda \in \Lambda$, be any Cauchy net in the bounded weak-star topology γ on B^*. Then $\{L_\lambda\}$ converges uniformly on compact subsets of B. Hence $L = \lim L_\lambda$ is a linear functional on B. But L is continuous, since it is continuous on each null sequence in B.

PROPOSITION. *Let B be a Banach space, and B^* its dual space. Suppose that in the weak-star topology, every sequentially closed subset of B^* is closed. Then B is finite-dimensional.*

Proof. We shall show that if B is infinite-dimensional, then there exists a subset X of B that is closed in the bounded weak-star topology but not weak-star-closed. Then X is sequentially closed. To construct the set X, supposing that B is infinite-dimensional, let $\{x_n\}$, $n = 1, 2, 3,...$ be a linearly independent sequence of vectors such that x_n tends to 0 in the topology of B. Then the set

$$U = \{x^* \in B^* : |\langle x^*, x_n\rangle| < 1, n = 1, 2, 3,...\}$$

defines a γ neighborhood of the origin that contains no α neighborhood of the origin. Otherwise, we could choose $y_1, y_2, ..., y_n$ in B such that $\{x^* \in B^* : |\langle x^*, y_i\rangle| < 1, i = 1, 2,..., n\} \subseteq U$. By the Hahn-Banach theorem, we may choose $x^* \in B^*$ so that $\langle x^*, y_j\rangle = 0$ for $j = 1, 2,..., n$, but $\langle x^*, x_m\rangle \neq 0$ for at least one m. Then λx^* lies in the weak-star neighborhood for all values of λ, but clearly cannot belong to U if λ is large enough. We choose X to be the complement of U.

Incidentally, this leads to another proof that $\alpha(G) \neq \beta(G)$ if G supports a nonconstant bounded analytic function. In that case, the bounded analytic functions separate the points of G and hence $B_H(G)$ is infinite-dimensional. This was shown in [19].

PROPOSITION. *Let B be a Banach space with dual B^*. The bounded weak-star topology of B^* coincides with the Mackey topology of B^* if and only if each weakly compact subset of B is norm-compact.*

Proof. The sufficiency of the condition is obvious from the characterizations of the topologies γ and m. Suppose now that $\gamma = m$ and that K is a weakly compact subset of B. Without loss of generality, we suppose that $0 \in K$. We define the absolute polar of K to be the set

$$K^0 = \{x^* \in B^* : |\langle x, x^* \rangle| \leqslant 1, \forall x \in K\}.$$

This is a basic Mackey neighborhood of the origin in B^*. Since K^0 is also a γ neighborhood of the origin, we must have $K^0 \subseteq C^0$ where C is compact. But C^{00} is the closed convex hull of C and the closed convex hull of a compact set is compact. Since a convex subset of B is closed if and only if it is weakly closed, we see that K must have been strongly compact to begin with, since it is weakly closed and contained in a compact set (see [13, p. 141]).

It was shown in [19] that $\beta(G)$ is not a metric space. The following result yields an alternative proof that applies to other spaces as well, such as the space of bounded harmonic functions.

PROPOSITION. *Let B be an infinite-dimensional Banach space. Then the bounded weak-star topology on the dual space B^* is not first-countable.*

Proof. Suppose that the origin has a countable base of γ neighborhoods. Then [14, p. 208] the γ topology may be given by a nonnegative paranorm ρ which satisfies (i) $\rho(x^*) = 0$ if and only if $x^* = 0$, (ii) $\rho(x^*) = \rho(-x^*)$, and (iii) $\rho(x^* + y^*) \leqslant \rho(x^*) + \rho(y^*)$. As we have just seen, B^* is complete in the γ topology. Consequently, B^* in the γ topology becomes a Fréchet space. Since the norm topology is stronger than the γ topology, the identity map I is continuous from B^* in the norm topology to B^* in the γ topology. By the open mapping theorem, the inverse map is also continuous, and hence the γ topology coincides with the norm topology. This implies that $\gamma = m$, the Mackey topology. By the preceding result, the weakly compact sets of B^* coincide with the strongly compact sets of B^*. In the case at hand, γ and m agree with the norm topology, and thus B is reflexive. However, the unit ball of a reflexive infinite-dimensional Banach space is weakly compact but not norm-compact. Thus, we contradict the assumption that γ be first-countable.

Problem. Characterize those Banach spaces for which weak and strong compactness coincide. To what extent do they resemble l^1 or spaces constructed from copies of l^1?

PROPOSITION. *Let B be an infinite-dimensional Banach space with dual space B^*. There exists a Hausdorff locally convex linear topology σ on B^* that is properly weaker than the weak-star topology, but that has the same convergent sequences, to the same limits.*

Proof. Choose a subspace $E \subsetneq B$ that is of the second category. Such subspaces may be obtained by selecting a countable subset $\{a_n\}$ from a Hamel base H for B. Then the linear span F_n of $H \setminus \{a_n\}$ must be of the second category for at least one n, since $B = \bigcup F_n$. The principle of uniform boundedness shows that the topology of convergence on finite subsets of E satisfies the conditions of the proposition.

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